

TOPICS IN ANALYSIS: DESCRIPTIVE SET THEORY

MATH 595

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MonWed 13:05–14:25

Location: BURN 1214

Originally arising as part of analysis, descriptive set theory (**DST**) is a peculiar combination of analysis/pointset topology (the infinite) and combinatorics (the finite), with little to no set theory in it¹. Its classical focus is the study of **definable** sets and functions in Polish spaces (e.g. \mathbb{R}^n , $\mathbb{N}^{\mathbb{N}}$, $L^p(\mathbb{R})$). Examples of definable sets include Borel sets, projections of Borel sets², the complements thereof, etc. A typical concern of DST is what **regularity properties** these sets have: are they measurable/Baire measurable, do they contain a Cantor set, etc? These are deep questions, often independent from ZFC, and at their heart lies the theory of **infinite games**, which we will study.



Cantor set

For the past 30 years, a major focus of DST has been the study of definable **equivalence relations** on Polish spaces. Such equivalence relations arise naturally all over mathematics since many mathematical objects (Riemann surfaces, Banach spaces, dynamical systems, etc.) can be parameterized as points in a Polish space. Classifying these points (e.g. Banach spaces) up to some notion of equivalence (e.g. isomorphism) means understanding the *Borel complexity* of this equivalence relation. DST provides a rigorous framework and tools for proving **classification results**, as well as their **impossibility**.

This study of equivalence relations is fueled by the study of **Polish groups** (including all countable groups) and their **actions**, as well as the combinatorics of **graphs** on Polish spaces, placing contemporary DST in the nexus of ergodic theory, topological dynamics, measured group theory, and graph combinatorics.

PREREQUISITES: [Math 454](#) (Analysis 3) or just basic knowledge of pointset topology and metric spaces. Having seen measure theory will help a lot but isn't necessary.

METHOD OF EVALUATION: There will be 4-5 homework assignments, with solutions presented at the board by the students. The grade will be loosely determined by these presentations and class participation.

COURSE MATERIAL: A. Tserunyan, *Introduction to Descriptive Set Theory*, lecture notes [[link](#)]

¹The name is a rather historical artifact.

²Lebesgue famously delayed the development of DST by a decade publishing a false proof that projections of Borel sets are Borel — they generally aren't!